3.2 Herbrand Structures

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Goal: prove unsatisfiability of formula in Skolem normal form.

In general, one would have to check all interpretations $I = (\mathcal{A}, \alpha, \beta)$ and prove that they do not satisfy the formula.

For formulas in Skolem normal form, the search space can be restricted significantly:

- β not needed, since formula is closed
- $\mathcal{A}$ can be fixed. We only have to consider $\mathcal{A} = \mathcal{T}(\Sigma)$ (i.e., the set of all ground terms)
- α_f can be fixed for all $f \in \Sigma$

  Only α_p is still open for $p \in \Delta$.

<u>Def 3.2.1</u> (Herbrand Structures, Jacques Herbrand)

Let (Σ, Δ) be a signature.

A <u>Herbrand structure</u> has the form $(\mathcal{T}(\Sigma), \alpha)$ where for all $f \in \Sigma_n$ with $n \in \mathbb{N}$, we have:

$$\alpha_f(t_1, \dots, t_n) = f(t_1, \dots, t_n)$$

If a Herbrand structure is a model of φ, then it is

Called a Herbrand model.

Goal: $\Phi \models \varphi$ should be checked automatically.

Φ: Prog. clauses $\{\varphi_1, \dots, \varphi_k\}$ $\qquad$ φ: Query

Solution: Instead, check whether

$$\underbrace{\varphi_1 \wedge \dots \wedge \varphi_k \wedge \neg \varphi}$$ is unsatisfiable

First step: Transform this formula to Skolem normal form $\forall X_1, \dots, X_n \; \psi$

ψ: quantifier-free, $\mathcal{V}(\psi) \subseteq \{X_1, \dots, X_n\}$

Advantage: We don't have to check all interpretations when proving unsatisfiability, but only Herbrand structures.

In a Herbrand structure S, every ground term is interpreted "as itself": $S(t) = t$ for all ground terms t

In a Herbrand structure, only α_p for $p \in \Delta$ is still "open", everything else is fixed.

Ex 3.2.2 Consider the signature (Σ, Δ) from Ex. 2.1.2. A Herbrand structure for (Σ, Δ) is $S = (\mathcal{T}(\Sigma), \alpha)$ with

$\alpha_n = n$ for all $n \in \mathbb{N}$

$\alpha_{monika} = monika$

$\vdots$

$\alpha_{date}(t_1, t_2, t_3) = date(t_1, t_2, t_3)$ for all $t_1, t_2, t_3 \in \mathcal{T}(\Sigma)$

$\alpha_{female} = \{monika, karin, \dots\}$

$\alpha_{male} = \{werner, klaus, \dots\}$

$\alpha_{human} = \mathcal{T}(\Sigma)$

$\alpha_{born} = \{(monika, date(25, 7, 1972)), (klaus, date(7, 5, 2017)), \dots\}$

Thm 3.2.3 (Herbrand Structures are Sufficient for Unsatisf.)
Let $\Phi \subseteq \mathcal{F}(\Sigma, \Delta, \mathcal{V})$ be a set of formulas in Skolem normal form. Then Φ is satisfiable iff Φ has a Herbrand model.

Proof: "$\Leftarrow$": trivial, since every Herbrand model of Φ is a model of Φ.

"$\Rightarrow$": Let $S = (\mathcal{A}, \alpha)$ be a model of Φ.

We now define a H-structure $S' = (\mathcal{T}(\Sigma), \alpha')$ such that S' is also a model of Φ.

For every $f \in \Sigma_n$, we define $\alpha'_f(t_1, .., t_n) = f(t_1, .., t_n)$.

For $p \in \Delta_0$, we define $\alpha'_p = \alpha_p$.

For $p \in \Delta_n$ with $n \geq 1$, then we define

$$\underbrace{(t_1, \dots, t_n) \in \alpha'_p}_{S' \models p(t_1, \dots, t_n)} \quad \text{iff} \quad \underbrace{(S(t_1), \dots, S(t_n)) \in \alpha_p}_{S \models p(t_1, \dots, t_n)}$$

Now we have to show that

$$S \models \varphi \quad \text{implies} \quad S' \models \varphi$$

holds for every formula φ in Skolem normal form.

So φ has the form $\forall X_1, \dots, X_n \ \psi$.

quantifier-free

We use induction on n.

<u>Ind Base</u>: $n = 0$

Thus: φ is a quantifier-free formula without variables.

We have $S' \models \varphi$ iff $S \models \varphi$. This can be proved easily by structural ind. on φ.

<u>Ind. Step</u>: $n > 0$

φ has the form $\forall X_1, \dots, X_n \ \psi$.

– φ has the form $\underbrace{\forall X_1, \dots, X_n \; \psi}_{\forall X_1 \forall X_2 \dots \forall X_{n-1} \forall X_n \; \psi}$.

Idea: Look at $\forall X_1, \dots, X_{n-1} \; \psi$.

This formula could contain the free variable X_n.

Let $S[\![X_n / a]\!]$ denote an interpretation that results from the structure S by using an arbitrary variable assignment $\beta[\![X_n / a]\!]$.

$$S \models \forall X_1, \dots, X_n \; \psi$$

iff $S[\![X_n / a]\!] \models \forall X_1, \dots, X_{n-1} \; \psi$ for all $a \in \mathcal{A}$

$\leadsto$ $S[\![X_n / S(t)]\!] \models \forall X_1, \dots, X_{n-1} \; \psi$ for all $t \in \mathcal{T}(\Sigma)$

iff $S \models \underbrace{\forall X_1, \dots, X_{n-1} \; \psi \, [X_n / t]}_{\text{in Skolem normal form. Thus: ind. hypothesis applicable}}$ for all $t \in \mathcal{T}(\Sigma)$

(by the subst. lemma 2.7.3)

$\leadsto$ $S' \models \forall X_1, \dots, X_{n-1} \; \psi [X_n / t]$ for all $t \in \mathcal{T}(\Sigma)$

(by the ind. hyp.)

iff $S'[\![X_n / \underbrace{S'(t)}_{t}]\!] \models \forall X_1, \dots, X_{n-1} \; \psi$ for all $t \in \mathcal{T}(\Sigma)$

since S' is a Herbrand structure

(by the subst. lemma 2.7.3)

iff $S' \models \forall X_1, \dots, X_n\ \varphi$ $\boxed{\text{☐}}$

<u>Ex 3.24</u> If Φ contains formulas that are not in Skolem NF, then Φ could be satisfiable, although it has no H-model.

$\Phi = \{p(a), \exists X \neg p(X)\}$ over the signature

$\Sigma = \Sigma_0 = \{a\}$, $\Delta = \Delta_1 = \{p\}$.

A model for Φ is $S = (\{0, 1\}, \alpha)$ with $\alpha_a = 0$ and $\alpha_p = \{0\}$.

This object is not represented by any ground term. Such "junk terms" are not needed for formulas in Skolem NF.

But Φ has no H-model, since a H-structure has the carrier set $\mathcal{T}(\Sigma) = \{a\}$.

Next Goal: Given a formula in Skolem NF, check whether it has a Herbrand model.

Idea: Reduce this problem from predicate logic to propositional logic.

More precisely, instead of proving unsatisfiability of a formula in Skolem NF, we prove unsatisf. of infinitely many formulas in propositional logic.

$\Rightarrow$ Replace all variables by all possible ground terms.

Def 325 (Herbrand Expansion of a Formula)
Let $\varphi \in \mathcal{F}(\Sigma, \Delta, \mathcal{V})$ be in Skolem NF, i.e.,
$\varphi = \forall X_1, \dots, X_n\ \psi$, where ψ is quantifier-free.
Then the Herbrand expansion of φ is defined as
$$E(\varphi) = \{ \psi[X_1/t_1, \dots, X_n/t_n] \mid t_1, \dots, t_n \in \mathcal{T}(\Sigma) \}.$$

Ex 326 Logic Program mother Of (ren, sus).
Query ?- mother Of (X, sus).

We have to check unsatisfiability of:

$$\text{motherOf}(ren, sus) \wedge \neg \exists X\ \text{motherOf}(X, sus)$$

$\Downarrow$ transf. to Skolem NF

$$\forall X \big(\text{motherOf}(ren, sus) \wedge \neg\, \text{motherOf}(X, sus) \big) = \varphi$$

$$E(\varphi) = \{\ mO(ren, sus) \wedge \neg mO(mon, sus),$$
unsatis-fiable $\rightarrow$ $$mO(ren, sus) \wedge \neg mO(ren, sus),$$
$$mO(ren, sus) \wedge \neg mO(date(1,2,3), sus),$$
$$\vdots$$
$$\}$$

$\leadsto E(\varphi)$ is unsatisfiable

Thm 327 (Satisfiability of Herbrand Expansion)
Let φ be a formula in Skolem NF.

Then φ is satisfiable iff $E(\varphi)$ is satisfiable.

Proof: $\forall X_1, \dots, X_n\ \psi$ is satisfiable

iff there exists a Herbrand structure S with

$S \models \forall X_1, \dots, X_n\ \psi$ (by Thm 3.2.3)

iff there exist a H-structure S with

$S[\![X_1/t_1, \dots, X_n/t_n]\!] \models \psi$ for all $t_1, \dots, t_n \in \mathcal{T}(\Sigma)$

iff there exists a H-structure S with

$S \models \psi[X_1/t_1, \dots, X_n/t_n]$ for all $t_1, \dots, t_n \in \mathcal{T}(\Sigma)$

(by the subst. lemma 2.2.3)

iff there exists a H-structure S with $S \models E(\varphi)$

iff $E(\varphi)$ is satisfiable (by Thm 3.2.3) $\boxtimes$

Formulas without variables correspond to propositional formulas:

Every atomic sub-formula $p(t_1, \dots, t_n)$ without variables can be seen as a propositional variable $V_{p(t_1, \dots, t_n)}$ which can be TRUE or FALSE.

Checking satisfiability means: Is there an assignment of truth values to all propositional variables such that the formulas become TRUE?

Ex 328: Reconsider $E(\varphi)$ from Ex 326:

It corresponds to

$\{ V_{mo(ren,sus)} \wedge \neg V_{mo(mon,sus)},$

This formula is already unsatisfiable → $V_{mO(ren,sus)} \wedge \neg V_{mO(ren,sus)}, \ldots \}$

For a finite set of propositional formulas / formulas without variables, satisfiability is decidable.

Idea: Generate the Herbrand-expansion $\{\psi_1, \psi_2, \ldots\}$ step by step.

First generate $\{\psi_1\}$. If it is unsat., stop with success.

Otherwise generate $\{\psi_1, \psi_2\}$. — " —

otherwise generate $\{\psi_1, \psi_2, \psi_3\}$. ...

Compactness Thm of Propositional Logic:

If a set Φ of formulas is unsatisfiable, then Φ has a finite subset which is unsatisfiable.

$\Rightarrow$ Our algorithm to check unsatisfiability is sound + complete.

Algorithm of Gilmore

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Input: $\varphi_1, \ldots, \varphi_k, \varphi \in \mathcal{F}(\Sigma, \Delta, \mathcal{V})$

Output: Determine whether $\{\varphi_1, \ldots, \varphi_k\} \models \varphi$.

1. Check unsat. of ξ
2. Check unsat. of ψ

$\vdots$

Alg of Gilmore

- Sound: if it returns "true", then $\{\varphi_1, \ldots, \varphi_k\} \models \varphi$ really holds
- Complete: if $\{\varphi_1, \ldots, \varphi_k\} \models \varphi$, then alg. terminates and returns "true"
- does not necessarily terminate if $\{\varphi_1, \ldots, \varphi_k\} \not\models \varphi$

is a semi-decision procedure for entailment

Unfortunately, it is very inefficient and not goaldirected $\Rightarrow$ we will improve it.